ON THE ANALYICAL INVESTIGATION OF NONSTATIONARY VIBRATORY SYSTEMS BY THE METHOD OF A FICTITIOUS OSCILLATOR

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Abstract

At use of the method of fictitious oscillator the analytical decisions for the description of the typical vibratory modes arising in systems with variable parameters are received. A dynamic effect caused by a sudden temporary change in the natural frequency of system (so called a "parametric impulse") as applied to problems of the dynamic of mechanisms will be discussed. Certain limitations connected with the fulfillment of the dynamic stability condition are resulted.

Key words: vibration, nonstationary system, dynamic stability.

Preliminary remarks. On studies of vibration of machines drives with cyclic mechanisms any coordinate in the absolute motion is a combination of the "gross" coordinate φ_0, realizing one degree of freedom of the absolutely rigid drive, and the "small" coordinates q_i, whose ensemble corresponds to H degrees of freedom of the vibratory system. Here, the set of differential equation is nonlinear, since the generalized coordinates and their derivatives enter as arguments of nonlinear functions. Using the linearization in the vicinity of the ideal phase by means of dynamically unimportant simplifications this system can be reduced to a set of linear differential equations with variable coefficients. In case all the nonlinear properties retained relative to the gross coordinate φ_0, and only small deformations q_i entered the corresponding expressions in a linear fashion.

The method of the so called "fictitious oscillator" (or "conditional oscillator") was presented in [1] and has received the further development in [2–6] and in this paper. The solution obtained by using this method as it comes from the analysis may, as a rule, be taken from a practical point of view to be sufficiently close to the exact one for the first approximation.

General information on the method of fictitious oscillator. A following second order differential equation serves as a mathematic model for the simplest systems with variable parameters

$$\ddot{q} + 2n(t)\dot{q} + k^2(t)q = w(t) \ . \quad (1)$$

Substituting into (3) an Euler expression

$$q = y\exp\left[-\int_0^t n(\xi)d\xi\right] \quad (2)$$

gives

$$\ddot{y} + p^2(t)y = W(t), \quad (3)$$

in which

$$p^2 = k^2 - n^2 - \dot{n};\ \ W = w(t)\exp\left[\int_0^t n\,dt\right].$$

The solution of the homogeneous equation, obtained from (3), will be sought in the form

$$y_* = B(t)\cos\Phi(t) \tag{4}$$

with one additional condition, $2\dot{B}\Omega + B\dot{\Omega} = 0$, where $\Omega = \dot{\Phi}$.

After substituting (4) into (3), one obtains after some transformations

$$y_* = A_0 \exp\left[-0{,}5(z - z_0)\right]\cos\left[p_0\int\limits_0^t e^z dt + \gamma\right], \tag{5}$$

where A_0, γ are amplitude and phase, calculated from the initial conditions; p_0 is an optional parameter with the dimension of a frequency; $z = \ln(\Omega / p_0)$ and $z_0 = z(0)$.

At the same time, the relation between the function $z(t)$ and variable frequency $p(t)$ will have the form of the following differential equation, responding to the particular " fictitious oscillator" with the excitation $2p^2(t)$

$$\ddot{z} - 0{,}5\dot{z}^2 + 2p_0^2 e^{2z} = 2p^2(t). \tag{6}$$

The particular solution of the differential equation (3) will have the form

$$y_{**} = \frac{1}{\sqrt{(\Omega(t))}}\int\limits_0^t \left[\frac{W(u)}{\sqrt{\Omega(u)}}\sin\int\limits_u^t \Omega(\xi)\,d\xi\right]du. \tag{7}$$

Allowed (2),(5),(7) the general solution of Eq.(1) is of the form

$$q = A_0 \exp\left[-\int\limits_0^t n(\xi)d\xi\right]\sqrt{\frac{\Omega(0)}{\Omega(t)}}\cos\left[\int\limits_0^t \Omega(\xi)d\xi + \gamma\right] + \\ + \frac{1}{\sqrt{\Omega(t)}}\int\limits_0^t \left\{\frac{w(u)}{\sqrt{\Omega(u)}}\exp\left[-\int\limits_0^t n(\xi)d\xi\right]\sin\left[\int\limits_u^t \Omega(\xi)d\xi\right]\right\}du \qquad (\Omega = e^z). \tag{8}$$

The expressions obtained establish the bounds between parametric vibrations for the initial system and the forced vibrations for the particular fictitious oscillator, which corresponds to the equation (6). The analogy described above is effective in representing vibrations of the initial system, and does not depend upon the function $p^2(t)$. Let us consider certain exact and approximated solutions of equation (6) for a set of characteristic cases.

Slow changes of $p^2(t)$. One assumes for this case, that the dynamic component of the equation (6) is small in relation to the static one:

$$\ddot{z} - 0{,}5\dot{z}^2 \ll 2p_0^2 e^{2z} \approx 2p^2(t). \tag{9}$$

At the same time it appears that $\Omega \approx p$ and that the solution (8) corresponds to the so called *VKB* approximation method [7].

The function $p^2(t)$ is a piece-wise constant. Using non-dimensional time $\theta = p_0 t$, we can rewrite equation (6) in the form

$$z'' - 0,5z'^2 + 2e^{2z} = 2\nu^2(\theta), \tag{10}$$

where

$$\nu = \frac{p}{p_0}; \quad z' = \frac{dz}{d\theta}; \quad z'' = \frac{d^2 z}{d\theta^2}.$$

A set of accurate solutions of Eq. (10) for $\nu =$ constant may be obtained after substituting $z'^2 = x$. Then

$$\frac{dx}{dz} - x = 4\left(\nu^2 - e^{2z}\right), \tag{11}$$

and from this

$$x = Ce^z - 4\left(\nu^2 - e^{2z}\right). \tag{12}$$

After finding the optional constant C we can deduce the following equation describing a phase trajectory

$$z' = \pm 2\sqrt{\left(0,25z_0'^2 + \nu^2 + e^{2z_0}\right)e^{z-z_0} - \left(\nu^2 + e^{2z}\right)}. \tag{13}$$

The function $p^2(t)$ subject to sudden change of value. If we will accept a particular solution of equation (9) in the form of a certain family of functions z with variable parameters $\beta_1 \cdots \beta_m$ then after substituting these into equation (9) we will obtain

$$p^2(t, \beta_1 \cdots \beta_m) = 0,5\ddot{z} - 0,25\dot{z}^2 + p_0^2 e^{2z}. \tag{14}$$

By a suitable choice of the variable parameters $\beta_1 \cdots \beta_m$ one can obtain "closeness" of the function $p^2(t)$ to the given function $p(t)$ for the discussed interval $[0, t_*]$, where the sudden changes of that function [2]. For determining the parameters β_i one can, in particular, use the method of quadratic approximation. In this case,

$$\int_0^{t_*} p^2 \frac{\partial p^2}{\partial \beta_k} dt = \int_0^{t_*} p^2 \frac{\partial p^2}{\partial \beta_k} dt. \qquad (k = 1, \ldots, m). \tag{15}$$

If values t_*, are small, then one can use conditions which exist at the boundaries of the interval $\left[0,t_*\right]$, since the influence of the kind of changes of $p^2\left(t\right)$ upon the solution is relatively negligible.

For the class of problems investigated here the family of functions $z=\ln\left(\beta_1\tau+1\right)+\beta_2\tau+\beta_3$ has merit.

The change of free vibration amplitude due to a parametric impulse. Let us assume that a step impulse occurs as

$$p=\begin{cases} p_0 & t<0 \quad \text{interval I} \\ p_1 & t<t<t_1 \quad \text{interval II} \\ p_0 & t>t_1 \quad \text{interval III.} \end{cases}$$

If the size of the time interval t_1 is close to the average size of the free vibration period, then the above representation of $p\left(t\right)$ as a piece-wise constant (square) function may be used not only for study of mechanisms with variable structure obtained for mechanisms with a continuous change in their free vibration frequency. For that case an exact solution of (13) will be used.

To begin, let us investigate the behavior of the system on the phase plane $z'\left(z\right)$, (Fig. 3). The most important thing here is to find those values of $z_{\min}$ for which the values of the amplitude $A_{\max}$ are maximal:

$$A_{\max}=A_0\exp\left[-0{,}5\left(z_{\min}-z_0\right)\right], \tag{16}$$

where $A_0=A\left(0\right)$.

Let us consider the case $\nu_1=p_1/p_0<1$ and the trajectory which during the first interval remains at $z_0=0;\quad z'=0$. This means a start from the origin of coordinates 0. In the second interval the phase trajectory follows the curve 3.
Let us find a point of that trajectory for which a switch from the second interval into the third one occurs. For that purpose we shall write

$$\theta-\theta_0=\int_{z_0}^{z}\frac{dz}{z'}. \tag{17}$$

Substituting into (17) the solution (13), we will obtain after integrating and transforming,

$$z=\ln\frac{2\nu^2\nu_0}{\nu^2+\nu_0^2+\left(\nu^2-\nu_0^2\right)\cos 2\nu\left(\theta-\theta_0\right)}, \tag{18}$$

where ν_0 corresponds to the value of ν at the end of the previous interval.

The value $z=z_1$, which is valid at the moment of switching, may now be found from the equation (18) by setting $\theta=\theta_1;\quad \nu_0=1$. We proceed then to substitute z_1 for z in equation (13), and thus find $z'\left(z_1\right)=z_1'$; in this way we determine the point N_i of the phase trajectory at which switching takes place. Two cases may be met. If $\theta_1 = p_0 t_1 < (\pi / 2\nu_1)$ there is a switching action at the point N_1; and as a result the function z will not reach its minimum during the second interval. If alternatively $\theta_1 > (\pi / 2\nu_1)$, the switchover is at the point N_2, and so for the second interval there will be observed a minimum value of $z_{\min} = 2\ln\nu_1$.

During the third interval a phase trajectory as curve 1 will be followed, as can be ascertained from equations (21) and (15) for initial conditions

$$\theta_0 = \theta_1; \quad z(\theta_1) = z_1; \quad z'(\theta_1) = z_1'; \quad \nu = 1; \quad \nu_0 = \nu_1.$$

The value of $z_{\min}$ can be calculated as the smaller root of the following quadratic equations, obtained for $z' = 0$ from (13):

$$e^{2z_{\min}} - e^{z_{\min}}\left[1 + \nu_1^2 + (1 - \nu_1^2)e^{-z_1}\right] + 1. \qquad (19)$$

The two points of the diagram M_1 and M_2' correspond to these values of $z_{\min}$. The case for which $\nu_1 > 1$ may be analyzed in a rather similar way. This time the phase trajectory for the interval follows the curve 2 from the point 0 second to the point N_3, and then will proceed according to the curves of the family 1. This case differs from that described above, since the minimum value of the function z for the second interval is equal to zero. This means that during the second interval $A \le A_0$. The amplitude in the third interval may however reach a great value, and this depends upon the value $z_{\min}$ (the point M_3). Using equations (15) and (21), we will determine now the ratio $\kappa = A_{\max} / A_0$, which is also called the amplitude increase coefficient:

$$\kappa = \left[\frac{\sqrt{H + \sqrt{(H^2 - 16\nu_1^4)}}}{2\nu_1}\right]^m, \qquad (20)$$

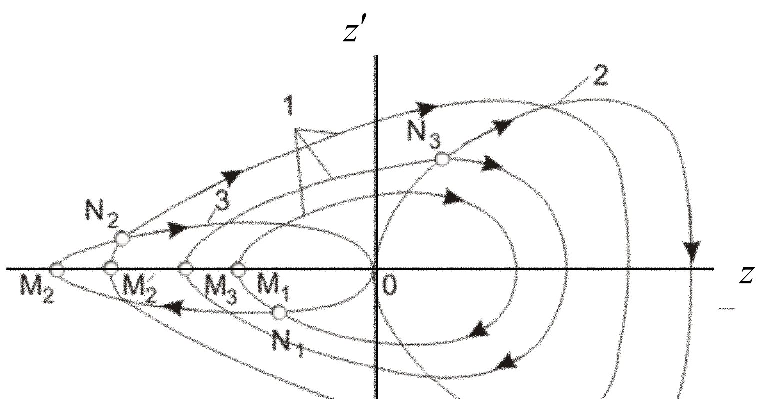


Fig.1

where

$$H=\left(1+\nu_1^2\right)^2-\left(1-\nu_1^2\right)^2\cos 2\nu_1\theta_1;\ \ m=\operatorname{sign}(\nu_1-1)$$

A plot of function $\kappa\left(\nu_1,\theta_1\right)$ is presented in Fig. 2

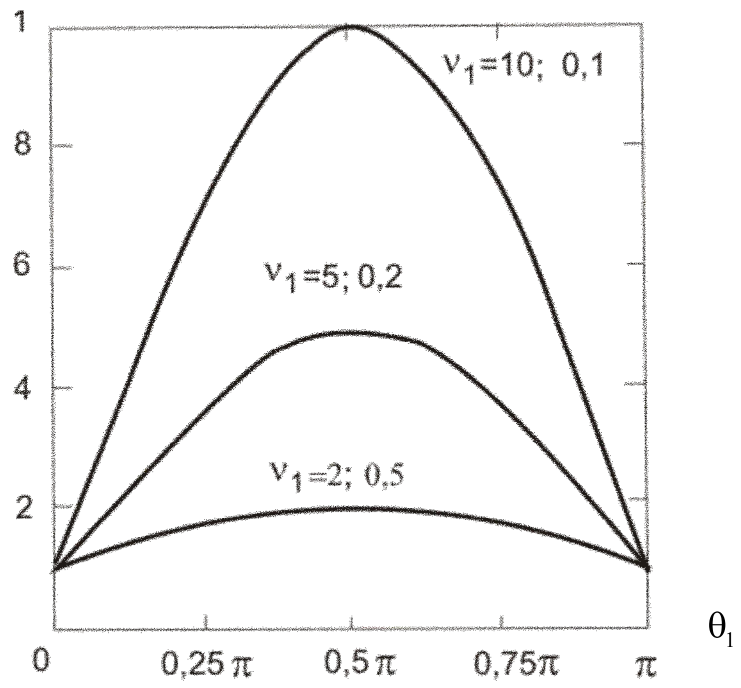


Fig.2

It is easy to show that $\kappa_{\max}=\nu_1^m$. By changing to an initial generalized coordinate q, one can write

$$q\le A_0\kappa\exp\left[-\int_0^t n(t)\,dt\right].$$

Harmonic pulsation of the function $p^2(t)$**.** Let us assume that $p^2=p_0^2\left[1-2\varepsilon\cos\left(\omega t+\gamma\right)\right]$. Such an assumption will convert equation (3) into a Mathieu equation. Considering the approximated solution of (6) to be in the form $z=a_0+a\cos\omega t$, and using the method of harmonic linearization, we will have

$$\left.\begin{aligned} a&=\frac{\varepsilon k_0^2(a)}{\left|k_0^2(a)-\left(\omega/2p_0\right)^2\right|}\cdot\frac{aI_0(2a)}{I_1(2a)};\\ a_0&=0,5\ln\frac{p_0^2+0,125\omega^2}{p_0^2I_0(2a)}.\end{aligned}\right\}\qquad(21)$$

Here

$$k_0^2(a) = \frac{I_1(2a)}{a\left[I_0(2a) - 0{,}5a\, I_1(2a)\right]}; \quad I_k(2a) = i^{-k} J_k(2ai); \qquad i = \sqrt{-1},$$

where J_k refers to Bessel's function with an imaginary argument $2ai$.

The value of $k_0(0)$ remains close to a constant even for relatively large values of a. For example, if $a \le 1$, then $1 \le k_0 \le 1{,}035$.

Linearization the equation of fictitious oscillator (case $|z| < 1$).

This case is very common in practical applications since it corresponds to the sufficiently large range of change of "natural" frequency, which may be estimated as $p_{\max} / p_{\min} \le 4$. In this case the fictitious oscillator possesses pronounced linear properties. So, for instance, when $|z| < 1$, the main frequency of free vibrations of the fictitious oscillator does not deviate more than 3,5% from the mean value. This oscillator allows, in the present case, for carrying out the linearization of the coefficients of Eq.(), after which it takes the from

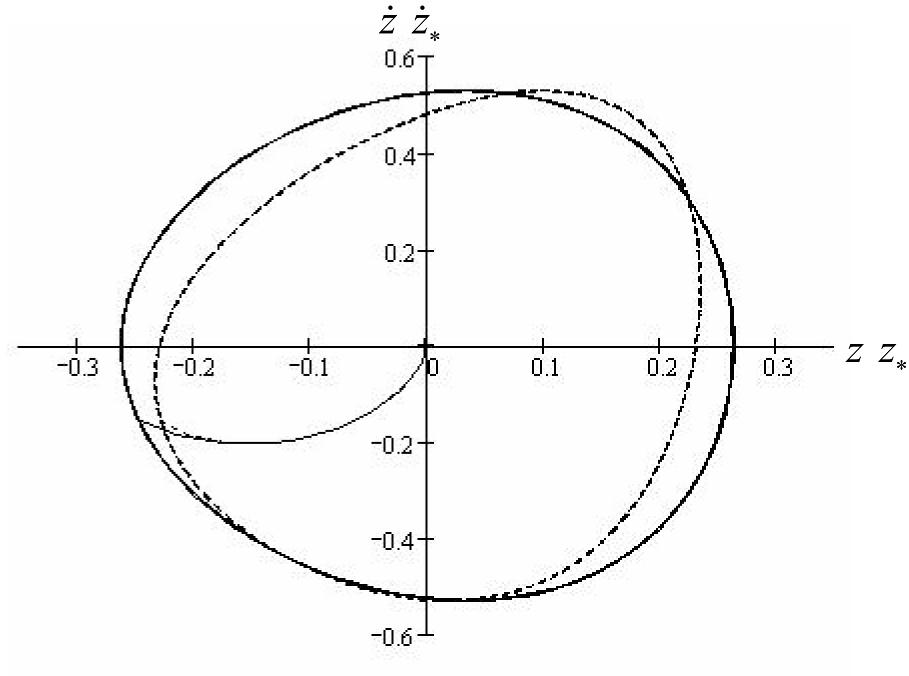
$$\ddot{z} + 4\overline{p}^2 z = 2(p^2 - \overline{p}^2), \qquad (22)$$

where $\overline{p}^2$ is the mean value of the function $p^2(t)$.

Eq. (22) has an exact analytical solution. Let us illustrate now the possibility of the parametrical excitation in the zone of the main parametric resonance. Suppose, for example, that pulsation of the function $p^2(t)$ takes place near the mean value $\overline{p}^2$. Pulsation frequency is ω, and $p^2 = \overline{p}^2(1 - \varepsilon\cos\omega t)$, where ε is the depth of the pulsation. Than from (22), $\ddot{z} + 4\overline{p}^2 z = -2\varepsilon\cos\omega t$.It is obvious that the fictitious oscillator resonates at $\omega = 2\overline{p}$, which corresponds to the main parametric resonance of the original system. The amplitude build-up of the fictitious oscillator is a necessary, though insufficient, condition for dynamic instability of the original system. On the other hand, it may be argued that the condition of a limited value of the variable amplitude in the first component of the solution (22) is sufficient (but not necessary) for dynamic stability. These conditions will be considered further.

On a phase plane (Fig.3) a good agreement of the exact solution $z(t)$ (continuous line) and the approached solution $z_*(t)$ (dashed line) is illustrated for a parametrical impulse (Fig.3,a) and a parametrical resonance (Fig 3,b).

a)



b)

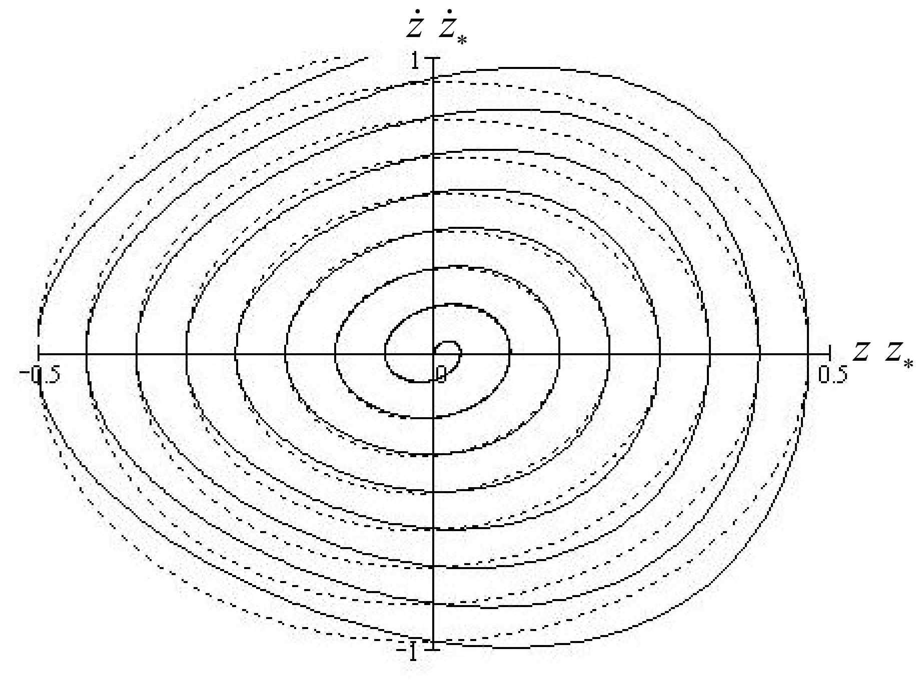


Fig.3

Analytical method of construction of solution for steady-state regimes

The equivalent differential equation with constant factors. The variable factor $n(t)$ in the differential equation (1) can be submitted as the sum dissipate and gyroscopic components $n_0 + n_1$.

We shall enter into consideration new variables: instead of time t we shall accept dimensionless time $\Phi = \bar{p}\int_0^t e^{z(\xi)}d\xi$, and instead of q – variable

$$\eta = q\exp\left[\int_0^t n_1(\xi)d\xi + 0,5z(t)\right] \qquad (23)$$

By using new coordinates (23) becomes the differential equation with constant factors:

$$\frac{d^2\eta}{d\Phi^2} + 2\delta\frac{d\eta}{d\Phi} + (1+\delta^2)\eta = L(\Phi), \qquad (24)$$

where $\delta = n_0/\bar{p} = \vartheta/(2\pi)$; $L = w\bar{p}^{-2}\exp\left[\int_0^t n_1(\xi)d\xi - 1,5z\right]$, ϑ is the logarithmic decrement; $\bar{p}$ is the mean value of the function $p(t)$.

Determination of the established modes. In the non resonance regimes, both function $n_1(t)$ and forced vibrations of the fictitious oscillator $z(t)$ are periodic functions of period τ. It is obvious that, in this case, function $L(\Phi)$ is also periodical, and the dimensionless value of the period of this function is equal to $\Phi(\tau) \approx \bar{p}\tau = 2\pi\bar{p}/\omega,$ where $\omega = 2\pi/\tau$. The dimensionless time Φ is a monotonously increasing function of time since $d\Phi/dt = \Omega(t) = \bar{p}\mathrm{e}^{z} > 0$.

Periodic function $L(\Phi)$ can be presented as a Fourier series for the argument Φ.

$$L(\Phi) = L_0 + \sum_j^{\infty}(L_{cj}\cos j\tilde{\omega}\Phi + L_{sj}\sin j\tilde{\omega}\Phi), \qquad (25)$$

where

$$\left.\begin{aligned}
&L_0 = \Phi^{-1}(\tau)\int_0^{\Phi(\tau)} L(\Phi)d\Phi = \bar{p}\Phi^{-1}(\tau)\int_0^{\tau} L(t)\mathrm{e}^{z(t)}dt;\\
&L_{cj} = 2\Phi^{-1}(\tau)\int_0^{\Phi(\tau)} L(\Phi)\cos j\tilde{\omega}\Phi d\Phi = 2\bar{p}\Phi^{-1}(\tau)\int_0^{\tau} L(t)\mathrm{e}^{z(t)}\cos(j\tilde{\omega}\Phi(t))dt;\\
&L_{sj} = 2\Phi^{-1}(\tau)\int_0^{\Phi(\tau)} L(\Phi)\sin j\tilde{\omega}\Phi d\Phi = 2\bar{p}\Phi^{-1}(\tau)\int_0^{\tau} L(t)\mathrm{e}^{z(t)}\sin(j\tilde{\omega}\Phi(t))dt;\\
&\tilde{\omega} = 2\pi/\Phi(\tau) \approx 2\pi/(\bar{p}\tau).
\end{aligned}\right\} \qquad (26)$$

After the decision the differential equation (24) at the account (25), (26) and considering dependence (2), after returning to variable q it is received

$$q = \exp\left[-\int_0^t n_1(\xi)d\xi - z(t)/2\right]\left\{L_0 + \sum_{j=1}^{\infty}\frac{L_j\sin[j\tilde{\omega}\Phi(t) + \gamma_j - \Delta_j]}{\sqrt{(1-j^2\tilde{\omega}^2)^2 + 4j^2\tilde{\omega}^2\delta^2}}\right\}, \qquad (27)$$

where $\delta = n_0 / \overline{p}$ $\left(\delta^2 \ll 1\right)$; $L_j = \sqrt{L_{cj}^2 + L_{sj}^2}$; $\sin\gamma_j = L_{cj} / L_j$; $\cos\gamma_j = L_{cj} / L_j$;
$\Delta_j = \mathrm{arctg}[2j\tilde{\omega}\delta / (1 - j^2\tilde{\omega}^2)]$ $\quad (j\tilde{\omega} \neq 2)$.

In practical cases, as it already has been mentioned, the condition $|z| < 1$ is usually satisfied. Than $z \approx z_*$ (see above), and

$$z \approx 2\sum_{j=1}^{j_{\max}} \frac{E_j}{4\overline{p}^2 - j^2\tilde{\omega}^2} \sin(j\tilde{\omega}t + \gamma_j - \Delta_j). \qquad (28)$$

At slow change of parameters $(j_{\max}\omega_0 \ll 2\overline{p})$ $e^z \approx p / \overline{p}$, and formulas (27), (28) accordingly become simpler.

This method is especially convenient at sufficiently smooth functions $p(t)$ and $w(t)$, when only a small number can be retained in a Fourier series.

Numerico-analytical method of construction a closed form solution.

For sharp changes of functions $p(t)$ and $w(t)$ the closed form solution is preferable. But in this case some difficulties emerge when we find a particular solution. In analysis of steady-state regimes, application of numerical methods of calculation often causes large cumulative errors due to a great number of integration steps. The numerico-analytical method of constructing the solution does not have this shortcoming. The method consists of several stages.

1. The numerical integration of the differential equation of the fictitious oscillator () with zero initial conditions. At this stage we calculate $z(t),\ z(\tau),\quad \Omega(\tau) = p\mathrm{e}^{z(\tau)},\ \Phi(\tau) = \overline{p}\int_0^{\tau} \mathrm{e}^z dt$.

2. The determination of the particular solution Y by the numerical integration of the differential Eq.(6) with zero initial conditions. At this stage we calculate $Y(\tau),\ \dot{Y}(\tau)$.

3. The determination of initial conditions corresponding to the steady-state regime: $q_0 = \mu D_0 \sin(\gamma^0),\ \dot{q}_0 = \mu D_0 \overline{p}\cos(\gamma^0)$, where $D_0 = \sqrt{A_0^2 + B_0^2}$,
$\gamma^0 = \gamma + \arcsin(\mu\mathrm{e}^{-9N}\sin 2\pi N)$; $A_0 = Y(\tau)$; $B_0 = [\dot{Y}(\tau) + 0{,}5\dot{z}(\tau)Y(0)]\Omega^{-1}(\tau)$;
$\mu = (1 - 2\mathrm{e}^{-9N}\cos 2\pi N + \mathrm{e}^{-2\pi N})^{-0{,}5}$; $\sin\gamma = A_0 / D_0$; $\cos\gamma = B_0 / D_0$; $N = \overline{p} / \omega$.

4. The numerical integration of the original differential equation with the thus defined initial conditions.

In this method, only certain intermediate functions in the limited interval of time are determined by numerical integration. Conditions of periodicity are inserted in the solution by an analytical method, specifically, by means of the method of the fictitious oscillator. The latter fact has a substantial effect on the solution accuracy and for obtaining effective engineering estimates as well.

If the parameters of the system are changing slowly, some simplifications of the method are possible. In this case there is no need to integrate the differential equation of the fictitious oscillator since $\Omega = \overline{p}\mathrm{e}^z \approx p$; $\Phi = \int_0^t p(\xi)\,d\xi$; $z = \ln p / \overline{p}$; $N = \overline{p} / \omega$.

Dynamic stability conditions.

In many mechanisms a change of the "natural" frequency takes place slowly. This does not exclude, however, the possibility of an increase in the vibration amplitude during certain intervals, caused by a local violation of the conditions for dynamic stability. According to (8) the amplitude of free and accompanying vibrations excited due to various pulse disturbances is changing in time proportional to the function

$$S = p(t)^{-0,5} \exp[-\int_0^t n(\xi) d\xi]. \quad (29)$$

It is easy to make sure that at constant parameters the function S is always decreasing and, therefore, that the free and accompanying vibrations are decaying. With changing parameters it may happen that $dS/dt > 0$, therefore the traditional decaying character of variation of these vibrations can be disturbed. This effect reveals a violation of the conditions of dynamic stability. In the similar case, the zone of the build-up is replaced by the zone of decay (Fig.4); therefore we do not experience the unlimited increase of the amplitudes typical, for example, to a parametric resonance.

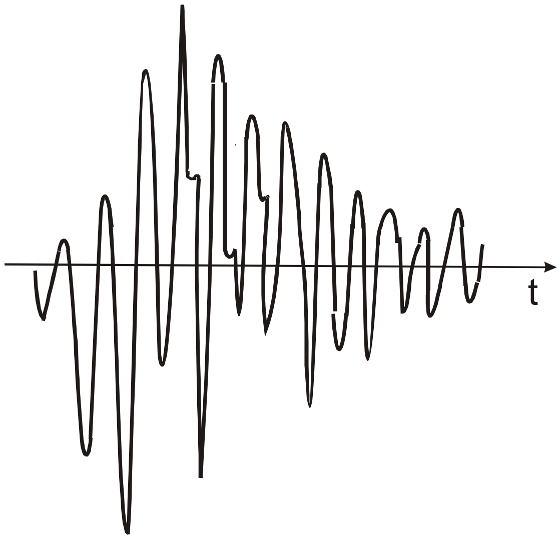


Fig.4

Nevertheless, under some unfavorable conditions the increase of amplitudes may become rather intensive. This calls for the dynamic synthesis to remove the possibility of the occurrence of such zones. It is easy to achieve by requiring $dS/dt < 0$. Using (29) we have

$$n + \frac{\dot{p}}{2p} > 0. \quad (30)$$

It is possible to show that condition (30) can also be obtained by the direct Lyapunov method and is, consequently, the sufficient condition for absolute stability [5]. Compliance with this condition removes also the possibility of the build-up in the zones of the main parametric resonances.

When parametric impulses are repeated periodically, then the maximum possible value of the amplitude for one period τ may reach a value as great as

$$A_1 = A_0 \exp\left[-\int_0^\tau n(t)dt\right]\prod_{i=1}^{s} \kappa_i, \quad (31)$$

where s is the number of parametric impulses occurring during this one period τ. Obviously, a sufficient condition necessary for suppressing a parametric excitation will be of the kind

$$\exp\left[-\int_0^{\tau} n(t)\right]\prod_{i=1}^{s}\kappa_i < 1. \qquad (31)$$

Conclusion. As suggested in [5,6], by using the quasinormal coordinates this method can be employed for nonstationary vibration systems with multi degrees of freedom.

The convenience of the method fictitious oscillator in the problems of machine dynamics is largely due to the fact that the adopted from of solution, which corresponds to this method, allows for use of formal procedures developed for systems with constant parameters. Thus, the procedure of analysis and calculation of vibratory systems becomes essentially a unified one.

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Поступила: 20.02.08.